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Musical logarithms in the seventeenth century: Descartes, Mercator, Newton

Benjamin Wardhaugh [1]

Hertford College, Oxford OX1 3BW, UK

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Abstract

This paper describes three previously little-studied sources from the 17th century, which reveal early uses of logarithms in the mathematical study of music. It describes the problem, which had existed since antiquity, of providing quantitative measures for the relationships between musical intervals when the latter were defined by identification with mathematical ratios; and it shows how this problem was solved by Descartes, Newton, and Nicolaus Mercator in the mid-17th century by using logarithms to provide "measures" of intervals, which could then be compared with one another. It discusses the composition and interrelationships of the manuscript sources for this work.

Résumé

Cet article présente trois sources du 17ième siècle—très peu étudiées jusqu'à présent—qui dévoilent une série d'usages précoces de logarithmes dans l'étude mathématique de la musique. Il fait ressortir le problème, déjà connu dans le monde antique, de la conception de mesures quantitatives pour correspondre aux relations entre les intervalles musicaux, lorsque ceux-ci étaient définis par leur identification avec des proportions mathématiques. L'article décrit alors la résolution de ce problème au milieu du 17ième siècle par Descartes, Newton et Nicolaus Mercator, qui se sont servi de logarithmes afin de fournir des "mesures" d'intervalles, mesures que l'on pouvait ensuite comparer l'une à l'autre. Cette description des sources est suivie d'une analyse de leur composition et des rapports qui les relient.
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In this paper I describe early uses of logarithms in the mathematical study of music, as revealed by three previously little-studied sources from the 17th century. In my recent doctoral work I have attempted to elucidate more fully the continuation of the Greek and medieval tradition of mathematical music theory by 17th-century writers, and the reader is referred to my thesis for more details of the context for the material I will describe here [Wardhaugh, 2006; see also Barker, 1989; Burnett et al., 1991; Carpenter, 1958; Christensen, 2002].

E-mail address: benjaminwardhaugh@fastmail.fm.
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1. Defining the problem: measuring and comparing musical intervals

Seventeenth-century Europe was a period of renewed interest in the use of mathematics to study music, a subject for which classical sources such as Ptolemy's *Harmonics* and the *Sectio canonis* attributed to Euclid were increasingly widely available in printed editions and translations [for example Meibom, 1652; Wallis, 1682; they are listed in full in Wardhaugh, 2006, 332–334]. The most common use of mathematics in this context was to discuss the details of musical tuning and to quantify musical pitch and musical intervals (the relationships between pairs of pitches, such as the octave and the perfect fifth). The ancient Greek tradition that this material was intended to continue most usually defined musical intervals by identifying them with mathematical ratios [see Barker, 1989, 5–8]. Two strings, otherwise identical, whose lengths form the ratio 2 : 1 will, when plucked or bowed, produce pitches an octave apart. The octave can therefore be correlated or even identified with the ratio 2 : 1. On closely similar grounds the musical intervals of the perfect fifth, perfect fourth, and tone were identified with the ratios 3 : 2, 4 : 3, and 9 : 8, respectively. In the sixteenth century the major third and minor third, with ratios 5 : 4 and 6 : 5, were added to this set. The strength with which the identification of intervals with ratios was made and the meaning that was assigned to it could vary. A numerological aspect was possible, although such musical writings of dedicated ancient "Pythagoreans" as ever existed are now lost [Aristotle, 1984, 985b23–986a21; Barker, 1989, 32; Burkert, 1972].

With these definitions an octave is equal to a fifth plus a fourth: that is, $2:1 = 3:2 \times 4:3$. The tone is equal to the difference between a fifth and a fourth: $9:8 = 3:2\,/\,4:3$. These examples show that the addition and subtraction of intervals correspond to the multiplication and division of ratios.

Musical intuition asks questions not just about the sums and differences of musical intervals, but also about their relative sizes. Here the identification with ratios causes problems, because ratios, being distinct from magnitudes, do not have "sizes." Although it is clear, for instance, that the octave is a larger interval than the tone, and it can be shown by simple trial and error that approximately six tones will fit into an octave, the question of exactly how these intervals are related to one another—exactly how many tones equal an octave, for example—was unanswerable with Greek mathematical techniques.

Several responses to this problem were possible. One was to divide the octave into a large number of equal parts and use these parts to approximate the sizes of the other intervals. This approach was first used in the Middle Ages, Boethius (c. 480–524) dividing the octave into 53 and Marchetto of Padua (early 14th century) into 31 parts [Boethius, 1989, III ch. 8; Marchetto de Padova, 1985; see Herlinger, 1981, 193]. It remained in use in the seventeenth century, when divisions of the octave were made in particular into 31 [Huygens, 1986; Rossi, 1666, 86; see Barbour, 1933, 292] and into 24 parts [Kircher, 1650, I, 208; Neidhardt, 1724, 31; Rossi, 1666, 102; see Barbour, 2004, 117]. To express an equal division of the octave numerically became much easier after the invention of decimal fractions late in the 16th century. To construct it geometrically was not possible until the rediscovery of the mesolabe, a device known to the Greeks but first described in modern times by Gioseffo Zarlino, writing in 1558 in a musical context. This device performed the construction of a large (in principle arbitrarily large) set of mean proportionals for a ratio expressed in lengths; that is, it enabled the finding of integral roots of a given ratio [Zarlino, 1558; see Barbour, 2004, 50].

Another response was to use the equal division of the octave not to approximate the musical intervals but to define them, abandoning the dependence of music theory on the ratios of small whole numbers. This seems to have been unacceptable to the majority of Greek music theorists, as indeed it remained unacceptable for many mathematicians interested in music theory in the 16th and 17th centuries (for example, see the citation from Mercator's *Rationes mathematicae* below). One reason for this was the numerological attractiveness of a music theory founded on a small set of small numbers: the basic Greek musical ratios could be found using only the set of numbers from 1 to 4, which the Pythagoreans called the *tetractys*; the expansion of the set of consonances in the Renaissance required the use of the numbers from 1 to 6, which their promoter Zarlino termed the *senario*. For much of the 17th century it was also widely believed that a mechanical justification existed for the requirement that all consonances should involve only ratios of small whole numbers, in terms of the coincidences of the pulses involved in pitched sounds [Benedetti, 1585; see Cohen, 1984, 75]. This belief (recognized as flawed only in the final third of the 17th century) slowed the acceptance of irrational ratios in music.

A crucial ancient Greek example of the use of equal divisions of the octave to define musical intervals without simple ratios was that of Aristoxenus, who divided the octave into 36 parts [Aristoxenus, 56.13ff; Barker, 1978; 1989, 168–169]. This procedure was sharply criticized by the influential Hellenistic writer Ptolemy, although "Aristoxenian" writings continued into the third and fourth centuries AD [Barker, 1989, 270–391 (Ptolemy), 3 (later Aristoxenians)].

Aristoxenus's overall approach to the study of music, more interested in the perceived experience of music than in mathematical constructions that were *a priori* supposed to correspond to it, ultimately became influential again after the rediscovery of his work and its circulation in manuscript in 16th-century Italy [Barker, 1989, 4–5; Fend, 1991].

A rare early modern example of the use of an equal octave division to redefine the musical intervals is that of Simon Stevin in c. 1605, a division into 12. Stevin's understanding of the relative status of rationals and irrationals was highly unusual and led him to assert that an equal division of the octave had more mathematical perfection than one involving rational ratios of small numbers [Cohen, 1984, 45–74; Stevin, 1955–1966]. This work remained unpublished until the 20th century.

Early sources that propose equal divisions of the octave are often hard to interpret and lack any clearly formed concept of approximation. The distinction between the two uses of equal octave divisions—to approximate the intervals or to redefine them—should therefore not be pressed.

A third response to the problem of the relative sizes of musical intervals was to use repeated subtraction of intervals in a procedure analogous to Euclid's algorithm, to find series of approximations for the relationship between two intervals. This procedure was used by Joseph Sauveur, working in Paris between 1700 and 1713, and by Conrad Henfling, a correspondent of Leibniz, active around 1706–1710, and there is indirect evidence that it could have been responsible for the popularity of certain equal divisions of the octave (into 12, 19, and 53) rather earlier [Bailhache, 1992; Sauveur, 1984; Wardhaugh, 2006, 71–80].

None of these responses directly addressed the fundamental problem of how to find the relative sizes of two musical intervals without ignoring or destroying their definitions in terms of simple ratios. To do this would have been to construct a "ratio of ratios," in the particular sense of finding that ratio that expresses how many times one ratio must be multiplied by itself to make another ratio. The concept in this sense was, as far as I know, first articulated by Nicole Oresme in the 14th century, who used it to support a refutation of the claims of astrology [Oresme, 1966] but lacked the mathematical means to compute specific examples of such ratios. The concept of a "ratio of ratios" in the simpler sense of—in modern terms—$(A/B)/(C/D)$ had existed probably since Pappus (fourth century AD), but this would not have been of use for the musical problem I have described.

The concept of a "ratio of ratios" in either sense, and indeed the notion that ratios could be compared or manipulated in the same way as magnitudes, was slow to gain acceptance. An example of the exclusion of such manipulations is provided by Isaac Barrow, who asserted in 1664 that ratios were not proper objects of mathematics at all: still less could they be subjected to the same operations as numbers or magnitudes [Barrow, 1734, no. XVI: pp. 293–312; see also Sasaki, 1985].

A late example of the calculation of ratios of ratios in the simpler sense is found in the posthumous *Opus geometricum posthumum ad mesolabium per rationum proportionalium novas proprietates* by the Belgian mathematician Grégoire de Saint-Vincent (1584–1667) (a different work from the same author's better-known *Opus geometricum quadraturae circuli et sectionum coni* of 1647). The work begins with a section on geometric ratios that, Gregory asserted, were "true quantities" [Saint-Vincent, 1668, proem [iii]].[2] Toward the end of Book I he discussed relationships between ratios, such as that a particular ratio of lines was the quintuple of another ratio of lines [Saint-Vincent, 1668, 13–17, 18–19]. He wrote repeatedly about the "ratio" of one ratio to another, but inspection of his arguments reveals that what he meant was what many writers would have called the subtraction of one ratio from another: when he wrote "the ratio of $A : B$ to $C : D$" he meant $(A : B)/(C : D)$. This reduced some of his theorems to trivialities.

2. Logarithms

The most successful discussion of ratios of ratios in the 17th century was also the earliest: Kepler's *Chilias logarithmorum* (1624), a very likely source for the slightly later idea of measuring musical ratios using logarithms. The work begins: "all ratios (*proportiones*) equal among themselves, whatever the differences between one pair of terms and another, are measured or expressed by the same quantity" [Kepler, 1624/1639, 280].[3] Kepler applied the idea of measuring a ratio using a quantity to various examples of continued proportion and to the division of given ratios into equal parts, continuing to discuss explicitly the measuring (*mensura*, *metior*) of ratios. He discussed how these

[2] "Verae quantitates."

[3] "Omnes proportiones inter se aequales, quacunque varietate binorum unius, et binorum alterius terminorum, eadem quantitate metiri seu exprimere."

measures behaved when ratios were compounded; he defined the terms "common measure" and commensurability and applied them to the measurement of ratios, going on to discuss the relationships between the measures of the various ratios produced by three or more given quantities [Kepler, 1624/1639, 284–287]. He showed how to use a given ratio (his example is 1000 : 999) to measure a large set of other ratios and showed for instance that 1000 : 997 was more than three times as large as 1000 : 999 [Kepler, 1624/1639, 291, 294]. (He came close here to the very elusive goal of a way of expressing the closeness of a given ratio to a given simpler ratio, for instance the closeness of 5001 : 4000 to 5 : 4, a concept that was explicitly formulated in a musical context only by Pietro Mengoli [Mengoli, 1670; Wardhaugh, 2007].)

Finally he defined logarithms as the measures of ratios: "the measure of any ratio between 1000 and a smaller number... expressed as a number, is placed by that smaller number in the [tables], and is called its logarithm, that is the number (*arithmos*) showing the ratio (*logos*) which it has to 1000..." [Kepler, 1624/1639, 297].[4] (So Kepler's logarithms decreased as the number in question increased.) He gave examples of the use of this concept to manipulate trigonometric ratios, and others for the behavior of the measures of ratios when ratios were divided in arithmetical or geometrical proportion.

In his *Complementum chiliadis* of the following year Kepler gave many further examples of the use of logarithms, some of which extended his idea of the measurement of ratios to their subdivision in specified proportions; that is, he constructed explicit ratios of ratios: "to divide a ratio of given terms into parts which are to each other in another given ratio." His first example was to divide 37 : 53 into two parts which had the ratio 5 : 2 to each other (the answer was to divide at, approximately, 47.83) [Kepler, 1625, 410–411].[5] He made rather little of this, but it was both sophisticated and novel. He gave an astronomical example of its use.

Although very occasionally Kepler's examples used sets of ratios that could have musical application, he did not make any explicit connection with music. This is surprising considering Kepler's interest in music and the fact that he approached music in *Harmonices mundi* using the manipulation of ratios [e.g., Kepler, 1624/1639, 290; see Kepler, 1619]. It is difficult to believe that Kepler never applied his measurement and division of ratios to musical ratios, but if such work exists it appears in neither *Chilias logarithmorum* nor *Harmonices mundi*.

A later development was that by Pietro Mengoli (1625–1686), who in his *Geometriae speciosae elementa* (1659) wrote about the use of logarithms to measure ratios: he developed in detail a theory of the ratios of ratios [Mengoli, 1659]. He followed the structure of the fifth book of Euclid's *Elements*, replacing each theorem on magnitudes with one on ratios, and each theorem on ratios with one on ratios of ratios. Massa Esteve [2003] explains in detail how Mengoli justified the use of logarithms as measures of ratios. Mengoli put this material to highly idiosyncratic musical use, which I have discussed elsewhere [Mengoli, 1670; Wardhaugh, 2006, 189–210; Wardhaugh, 2007].

It should be clear from what has been said that the use of logarithms could solve the problem of comparing musical intervals without resorting to approximations or redefinitions of the intervals. In particular, logarithms facilitated the production of a representation of pitch, either visual or numerical, in which equal musical intervals would occupy equal spaces. Since the identity of a musical interval is unaffected by its position in the scale, such a quantification was desirable, a desire that was not satisfied by the conventional quantification of pitch, which identified it with the length of a sounding string.

Because of the additive property of logarithms, that $\log(ab) = \log(a) + \log(b)$, the addition of logarithms corresponds to the multiplication of ratios and therefore to the addition of musical intervals:

$$\text{a fifth plus a fourth equals an octave,}$$
$$\text{i.e., } 3:2 \times 4:3 = 2:1.$$

In logarithms:

$$\log(3:2 \times 4:3) = \log(2:1),$$
$$\log(3:2) + \log(4:3) = \log(2:1).$$

[4] "Mensura cujuslibet proportionis inter 1000. et numerum eo minorem... expressa numero, apponatur ad hunc numerum minorem in Chiliade, dicaturque LOGARITHMUS ejus, hoc est, numerus (*arithmos*) indicans proportionem (*logon*) quam habet ad 1000...."

[5] "Proportionem cum terminis suis datam secare in partes, quae sint ad invicem in alia proportione proposita."

It follows that we can substitute the addition of logarithms for the addition of intervals. Other operations correspond in a similar manner: the subtraction of logarithms corresponds to the subtraction of intervals; the multiplication of a logarithm by a number corresponds to the multiplication of an interval by that number. Logarithms provide a "measure" for intervals, and, crucially, those measures can be compared with one another and can be divided by one another. So the relative size of two intervals can be quantified by the ratio of their logarithmic "measures." For example, the relative size of the fifth and the octave is given by the ratio of their logarithmic sizes:

$$\log(3:2):\log(2:1) \approx 176:301.$$

Although isolated musical calculations of this kind were performed by others [see Barbour, 1940], René Descartes was the first to produce a diagram of musical pitch, a complete representation of the musical octave, apparently using logarithms, and Nicolaus Mercator and Isaac Newton both followed him. Unfortunately, none of the three published the details of this mathematical work. In the remainder of this article I will discuss the musical mathematics of these three authors as it is extant in a number of manuscript sources.

3. Descartes

Descartes' *Compendium musicae*, as published posthumously in 1650, contains four circular diagrams. It was written in 1618 and, uniquely among Descartes' works, prepublication manuscript copies survive, although none is Descartes' autograph. There are four. The earliest is a rather rough copy made in his journal by Isaac Beeckman in about 1627–1628: this copy apparently does not contain the circular diagrams [Middelburg (Netherlands), Provincial Library of Zeeland, MS. "Journal of Beeckman", ff. 163r–168v]. A copy made for Constantijn Huygens in 1635 has four diagrams similar to that in Fig. 1, carefully and accurately drawn [Leiden, University Library, MS. Hug. 29 a]. These diagrams also appear in the copy attributed to the Dutch mathematician Frans van Schooten of 1641 [Groningen, University Library, MS. 108, ff. 60r–83v]; while the English mathematician John Pell, making a copy in 1650, left spaces for all of the diagrams but drew only one [London, British Library, Add MS. 4388, ff. 70r–83v; see also Malcolm and Stedall, 2005, 569–574; Otegem, 1999].

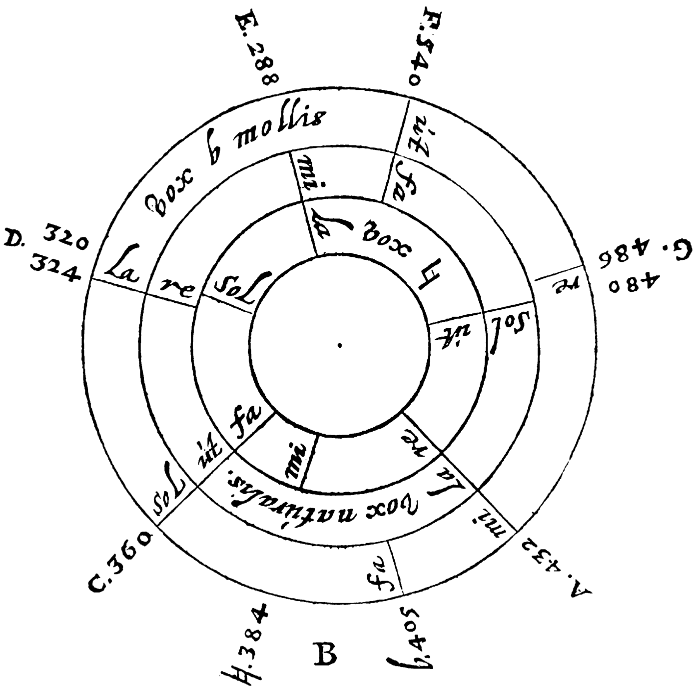

Fig. 1. Circular pitch diagram. Descartes, *Musicae compendium* (1650), 35. By permission of the Bodleian Library, University of Oxford (shelfmark: 70 d. 21, 4).

The published versions of the treatise most relevant for my purpose are the first Latin version (Utrecht, 1650) and the French translation of 1668 [Descartes, 1650, 1668]. Each contains tolerably accurate versions of the diagrams: Fig. 1 reproduces one of those from the first Latin edition. For the second Latin edition (1656) the diagrams were simply reprinted without change from the 1650 edition, and the 1653 English version, as I will discuss, replaced them with crucially different diagrams [Descartes, 1653, 1656]. (In Adam and Tannery's edition of Descartes' works the diagrams are redrawn and distorted quite badly, but Buzon's edition of the *Compendium* very usefully reproduces in facsimile those of both the 1650 and 1668 editions [Descartes, 1897–1909, X, 104, 118, 120; Descartes, 1987].)

In these diagrams the circumference of the circle represents one octave of pitch. But, crucially, within that octave equal musical intervals have equal sizes, to an accuracy usually of within one or two degrees. Unlike those who divided the octave into equal parts by arithmetical or geometrical methods, Descartes actually created here a scale of pitch in which intervals defined by ratios retained the relative sizes specified by those definitions.

Descartes took the logarithms of the intervals' ratios and scaled them so that the octave $(2:1)$ is represented by the whole circumference of the circle (360°). An interval with ratio A is represented by an angle of $360 \cdot \log A/\log 2$. For example, the fifth $(3:2)$ has size $360 \cdot \log(3/2)/\log 2 \approx 211°$.[6]

This is close to the use of the logarithm as a function to transform one line into another, a use which is very striking at such an early date. Since the first logarithm tables were published only in 1614 [Napier, 1614], and also because the earliest copy of Descartes' treatise lacks the diagrams, it seems quite probable that the diagrams did not appear in the original 1618 version of the treatise, which Beeckman copied in 1627–1628.

The Beeckman and Huygens manuscripts are both thought by Otegem to be first-generation copies from an original by Descartes, and I think it likely that the diagrams originated in about 1635 (the date of the Huygens copy) with work done in collaboration between Constantijn Huygens and Descartes.[7] It also seems probable, although this cannot be proved, that Kepler's work on logarithms provided some stimulus for their use to produce this representation of pitch. The van Schooten and Pell manuscripts are second-generation copies deriving from the Huygens manuscript. The first printed edition was based, probably indirectly, on the van Schooten manuscript. At each stage in this transmission there is detectable deterioration in the accuracy of the diagrams.

The English version of Descartes' *Compendium* that appeared in 1653 has various interesting features. It does not name the translator, but the records of the Stationers' Company of London state that he was Walter Charleton, an English atomist whose own theory of sound and musical experiments I have discussed in [Wardhaugh, 2006, 162–164, 223–226]. A lengthy set of "animadversions"—actually a point-by-point refutation of Descartes—follows the translation, and there is evidence that this was written by William Brouncker, the future President of the Royal Society [see Descartes, 1987, 37–39].

The diagrams are curious: an example is given in Fig. 2. Although it is superficially similar to the corresponding diagram in the original Latin edition, detailed measurement of the angles' sizes reveals that the logarithmic scale of pitch is absent. Instead, the circumference of the circle represents part of a musical string, and the positions on the circumference marked out by the various radii correspond to the placement of the fingers on that string to produce the notes indicated. Equal musical intervals do not occupy equal spaces in this representation. The circular form of the diagram and the appearance of three different musical scales in concentric rings are clearly modeled on Descartes' diagrams, but the numerical content is that of much more traditional representations of pitch, which represented stringed instruments and indicated the positions of the fingers on the strings used to produce different notes: one of many examples appears in Brouncker's "animadversions" themselves [Descartes, 1653, 66–67, reproduced in Gouk, 1999, 142–143].

This transformation was repeated in each of the diagrams. It is impossible to be sure whether this was a deliberate replacement of a more complex representation of pitch with a simpler one, or whether the translator and editor simply failed to understand the content of the original diagrams. The latter is suggested by a negative comment on the accuracy of the original diagrams in the translation's preface [Descartes, 1653, b2r–v].

[6] It is not absolutely impossible to make diagrams of this kind accurately without using logarithms. The equal division of the octave into 53 can be used to provide a rather close approximation to the scale that is represented here. The details of the angle sizes in these diagrams do not really suggest that such an approximation was used—angles smaller than $1/53$ of a circle appear—although they do not absolutely rule it out.

[7] I am grateful to Matthijs van Otegem for additional information about this point. I have not examined the text of Beeckman's copy, which would probably shed light on this conjecture.

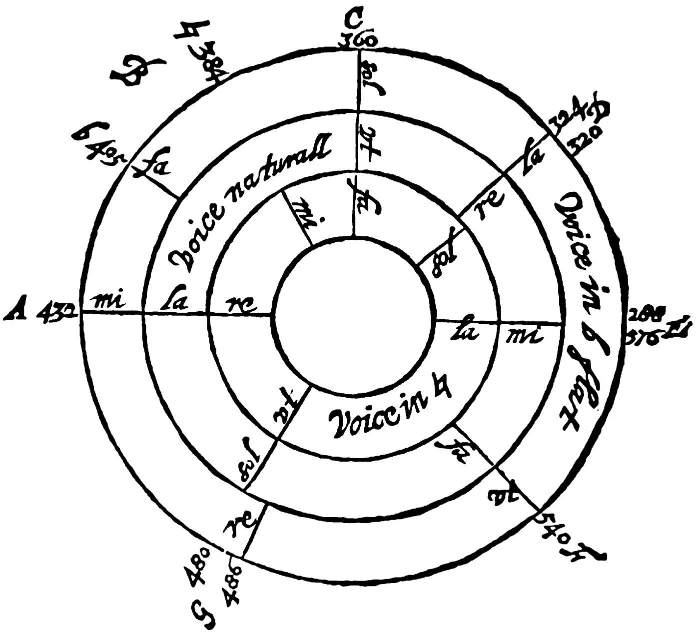

Fig. 2. Descartes' representation in the English translation (Walter Charleton, Trans., William Brouncker, Ed.), *Renatus Des-Cartes Excellent Compendium of Musick*, 35. By permission of the Bodleian Library, University of Oxford (shelfmark: 4^{o} R4 Art B5, 3).

Although the English translator thus undervalued them, we should not by contrast overstate what the author of the original diagrams in *Compendium musicae* achieved. Aristoxenus apparently drew or at least envisaged linear diagrams of pitch with equal intervals of equal sizes; and any theorist who tried to compare different musical intervals thereby revealed that he held intuitively a concept of pitch similar to that demonstrated here. Stevin had expressed the string lengths for the equal division of the octave in exact numerical form in c. 1605: a step toward the logarithmic scale of pitch. But, as far as I know, these diagrams are the earliest accurate representation of the relative sizes of intervals defined by ratios.

4. Mercator

4.1. The manuscripts

Nicolaus Mercator (1620–1687) never published on music, but several manuscripts attest to his interest in the subject, and in fact his work on the problem of a musical "common measure" is the most sophisticated I have seen. I am aware of seven relevant texts: namely a notebook, a partial write-up of that notebook, and one very brief note containing draft material for a treatise on musical ratios; a printed prospectus for that treatise; two draft versions of the treatise, the less complete of which exists in two copies; and finally a discussion of the musical theories of John Birchensha (fl. 1661–1675), apparently unrelated to the other texts. Only the printed prospectus can be dated with certainty, but I shall present briefly a tentative chronology of the material before discussing in detail those aspects of its technical contents that interest me.

The draft version of Mercator's treatise on musical ratios, in English, appears in two closely similar copies in Oxford, Bodleian Library, MS Aubrey 25, ff. 1–25 and London, Guildhall Library, MS. 51757 21, ff. 139–153. I will refer to this treatise as "Musica," from Aubrey's title page, although Aubrey also used that title for another text. Aubrey ascribed the text to Mercator.

"Musica," which I believe represents Mercator's earliest thinking on music, has major sections on the consonances and on the arithmetic of proportions using ratios or using logarithms, plus an introductory list of definitions, a discussion of the division of a string, with diagrams, and an elaborate procedure for dividing one superparticular ratio (that is, of the form $n + 1 : n$) into two others. It uses this procedure to construct a scale, by repeated division of the octave into smaller superparticular ratios: this construction is not found in the other texts, but could be construed as an attempt to systematize a procedure found occasionally in Ptolemy [Barker, 1989, 260]. In [1694], William Holder would make much use of the same construction in his work on mathematical music.

The Guildhall copy of "Musica" is part of a collection of manuscripts believed to have belonged to Robert Hooke and possibly in the hand of one of his assistants. (Its second and final leaves, ff. 153 and 139a, were apparently exchanged prior to foliation.) Hooke's diary records that he had "Mercator's Music" copied on 23 October 1676: in the absence of any other candidate I assume that the Guildhall "Musica" is the copy to which he refers [Hooke, 1935/1968, 254]. I have not investigated in detail the differences of spelling and punctuation between the two texts of "Musica," which might possibly allow us to determine whether these are independent copies of a lost original, or whether one is a copy of the other.

Aubrey gave his copy the date 1673 and stated on the title page that "the Original copie was lost at Paris." Mercator came to England in 1654, was in Paris from 1655 to 1657, and then lived in London from 1657 until 1682, so this suggests—although other explanations are possible—that that original was written at the latest by 1657, making 1673 the date of copying, not of composition [Scriba, 2004]. It is difficult to explain the production of this text in English at any date, since, as with Descartes' *Compendium*, it is hard to envisage a reader interested in the contents and likely to understand them but unable to read Latin. Nonetheless the simplest assumption is surely that the English text was made during Mercator's first period in England, that is, 1654–1655.

This early date can be tentatively corroborated by the fact that "Musica" shows signs of having been composed, or translated from Latin, by a non-native speaker of English, whom I take to be Mercator himself. It consistently uses "reason" where "ratio" (or, in seventeenth-century English, "ration") would be expected, and never uses common contractions such as "y^e^" or "w^ch^." The Guildhall manuscript of "Musica" does use some of these contractions, though it retains "reason."

Since "Musica" is the least developed of the texts we have, I suspect that it is in fact a translation of a Latin original, which I take to be the "original" that was lost at Paris, and which would also have been the earliest of Mercator's musical writings. This makes it hard to explain its circulation in the 1670s, which is surprising in any case since the text (in both copies) breaks off abruptly after the statement that a whole scale is about to be given on the basis of the preceding calculations.

A notebook with musical material that can be attributed to Mercator [Oxford, Christ Church Library, MS. 1130] ("Notebook"), contains draft material in Latin on various uses of ratios, with musical topics overwhelmingly predominating. Although the material is clearly organized and carefully thought out, presentation has not been an issue. The book has been used from both ends, and the notes on nonmusical topics intrude into the musical material apparently haphazardly. It is undated and contains rather more material than "Musica." The text begins with the arithmetic of ratios, including the use of logarithms, followed by the derivation of the musical intervals. A large section follows on the finding of a common measure for musical ratios by the division of the octave into fifty-three equal parts. This is presented as a numerically simpler equivalent to the use of logarithms. It precedes a discussion of the scale, which is expanded to include the finding of the sizes of all the diatonic intervals in terms of this common measure. Material on musical modes (scale types) precedes a discussion of temperament and the division of the musical string: finally there is a section detailing six rules for composition, dealing with rather general notions such as "formality," "variety," and "plenitude." Compared with "Musica" it lacks the introductory definitions and the procedure for the division of superparticular ratios. Small sections are included on how to put a ratio into its lowest terms and on the use of musical notation, particularly clef signs. On nonmusical uses of the arithmetic of ratios it mentions Kepler's third law of planetary motion, monetary calculation, the calendar, and geometry. It seems intended as the basis for a treatise of much larger scope than "Musica," and I believe that it is of later composition than "Musica."

[Oxford, Christ Church Library, MS. 1187 D 14], which I will call "Theoria musices" from its title page, is essentially a neater version of the musical sections of "Notebook," with some extra material added, and some of the mathematical details omitted. In its first pages it has the appearance of a fair copy, but this is not maintained throughout the manuscript, which gradually becomes less neat and less clearly organized, and eventually begins to include arithmetical working. No nonmusical material is found here, but otherwise this text contains nearly the same material as "Notebook," in nearly the same order. The main difference is one of ordering: the section on temperament and the division of the string is now placed before the discussion of the modes. All that is omitted compared with "Notebook" is the small sections on reduction of ratios, and the use of clefs and other notation and the notes on nonmusical uses of ratio. All that is new is a brief introduction on the nature of sound and an enumeration of the species of ratios. The sections on the scale, temperament, and modes are rather more brief, and the section on composition is somewhat expanded. I believe that this manuscript is later than the “Notebook” and represents a partial write-up of its musical material.

The list of topics in “Notebook” corresponds quite closely to that of *Rationes mathematicae*, a pamphlet published by Mercator in Copenhagen in 1653. This reads like a prospectus for an extensive work on various uses of mathematical ratios, including their musical use [Mercator, 1653]. I suggest that both “Notebook” and “Theoria musices” represent the collection and drafting of material for the project described in *Rationes*. Mercator left Copenhagen for England late in 1654 because of the plague: his proposed work on ratios never appeared [Scriba, 2004]. The only known copy of *Rationes* survives in Paris, suggesting that Mercator kept the plan warm at least until his arrival there in 1655.[8]

A very brief note of some questions requiring consideration appears on a single sheet in Mercator’s hand among the Hartlib papers in Sheffield University Library [Greengrass and Leslie, 1995/2002, 56/1/152A–B]. Here, as in *Rationes*, the questions of dividing the musical scale and dividing astronomical time are juxtaposed, with the suggestion that the appropriate method of solution is one based on ratios. This seems likely to relate to the project described in *Rationes*, although it is undated. (It must date from before Hartlib’s death in March 1662.)

A second treatise, in Oxford, Bodleian Library, MS. Aubrey 25, ff. 32–43 (“Of music”), not obviously incomplete, treats much of the same matter as “Musica,” “Notebook,” and “Theoria musices.” Though not attributed in Aubrey’s copy, its close relationship with the other Mercator material, as well as its juxtaposition in MS. Aubrey 25 with Mercator’s “Musica,” makes it apparently another work of his. It is on the whole briefer than the other texts, and, while it is conceivable that it is another early treatise like “Musica,” its clearer organization and more developed contents argue that it is a later revision of the musical material. It lacks the translation characteristics of “Musica” and shares certain quirks (the use of “th’other,” in particular) with Mercator’s other original writings in English, which also argues that its English form may be a little later than Mercator’s initial move to England. Aubrey dated his copy 1672, which could in this case be the date either of copying or of composition.

“Of music” is more clearly structured than “Musica,” “Notebook,” or “Theoria musices”: like “Theoria musices” and “Musica” it uses numbered sections, but unlike them it does not obscure this numbering with sub- and subsubnumberings, or with sections of widely varying lengths. It discusses the following topics: the consonances; the arithmetic of ratios either simply or using logarithms; the scale and the intervals between consecutive pitches, and their ratios; the division of the string and the “mean tone” scale; and the 12 modes and their natures and transpositions. It is the smallest in terms of content: it contains only material also found in one or more of “Musica,” “Notebook,” or “Theoria musices.”

Since Mercator published only in Latin, it seems relatively unlikely that this English text, if it is Mercator’s last writing on music, reflects a renewed plan to publish on musical ratios. I offer no suggestion as to why Aubrey, as far as we know, obtained copies only of this and of “Musica,” not of the more extensive, if messier, “Theoria musices”; nor as to why Hooke seems to have had only “Musica” copied.

Finally there are a few sheets in Mercator’s hand among the papers of the English mathematician John Pell (1611–1685) in the British Library, discussing the tuning theory of John Birchensha [London, British Library, Add MS. 4388, ff. 39–44] (for Birchensha see Field and Wardhaugh, 2008; Wardhaugh, 2006, 273–292). These are unsigned and undated, but read like part of a letter. The content of the text is consistent with Mercator’s other musical writings, so that he may well have been its author as well as its copyist. Mercator and Pell had become acquainted in the 1640s [Malcolm and Stedall, 2005, 106, n. 19; Scriba, 2004]; Birchensha is visible as a music theorist in other sources from 1661 to 1675. Some of Pell’s calculations (which I suspect may have been carried out as a result of a request for help by Birchensha) are dated 1665, and it seems possible that Pell had asked for Mercator’s opinion. This would also provide a plausible reason why Mercator revisited his musical writings after his return to London, to make “Of music.” Mercator may even have written “Of music” for John Pell, although the absence of logarithms from Pell’s own musical calculations would then be surprising.

Although I have presented a tentative sketch of their chronological relationship, the relative dates of these texts are in fact a puzzle for which no completely satisfactory solution seems forthcoming. The development of the material suggests to me that “Musica” was Mercator’s first word on music and “Of music” his last, with the “Notebook” and “Theoria musices” falling in between them. But this judgement is necessarily somewhat subjective, and on this or any

[8] Paris, Bibliothèque Nationale de France, shelfmark V-6398 (6).

account it is hard to explain the existence of "Musica" in English and its circulation, apparently in an incomplete state, in the 1670s, while "Notebook" and "Theoria musices" (which as far as we know did not circulate) seem related to *Rationes*, published 20 years earlier.

My suggestion, then, is this. At some point before 1653 Mercator wrote "Musica" in Latin and began to plan a treatise on ratios in general, producing the "Notebook" and a partial write-up of its musical part, "Theoria musices," both of which contained more mathematical development than "Musica." In 1653 he published the prospectus, *Rationes*, but his plans for a treatise were disrupted by the plague in 1654 and his move to England. In England between 1654 and 1655 he (partially) translated "Musica" into slightly unidiomatic English. In Paris between 1655 and 1657 he lost the Latin original of "Musica" and left at least one copy of *Rationes*, perhaps revisiting his plan to write a large treatise on ratios, but with no result. Later, back in England, probably between 1662 and 1675, he supplied John Pell with a written opinion on a (now lost) description of John Birchensha's theory of music. This, or something else, stimulated Mercator to make one more, English, version of his music treatise, "Of music." This and the translation of "Musica" circulated between at least 1672 and 1676, but Mercator did not as far as we know write on the subject again.

Hardly any of this is certain, and much of it may well be wrong. It is not hard to find alternative explanations, including simpler ones that do not rely on a lost Latin original for "Musica." But the relative state of development of the mathematics in the various texts, the knowledge that *something* was "lost at Paris," and the rather close correspondence between "Notebook" and *Rationes* all point toward a story of this general shape.

I turn now to the contents of these texts. For me their chief points of interest are the sections on arithmetic of ratios with or without logarithms, the common measure, and the use of logarithms or the common measure to deal with actual questions of musical tuning or temperament.

4.2. *Their mathematics*

Each of Mercator's texts has a section on the arithmetic of ratios, but that in "Musica" is the clearest and the most clearly distinguished from the use of logarithms. Its description of how to add ratios gives a flavor of the whole:

> Multiply the antecedent of one reason by the antecedent of th'other reason; the product is the antecedent of compound reason. So the consequent of one reason multiplyed by the consequent of th'other; the product is the consequent of the compound reason [Oxford, Bodleian Library, MS Aubrey 25, f. 4r].[9]

Subtraction is similarly described, performed by multiplying antecedents by consequents (cross multiplication). In both of these cases Mercator notes an alternative method using fractions:

> The modern way, by writing downe the reasons, as fraction[s], though ⟨they are⟩ quite differing things.
> The reason 2 to 3, is written thus $2/3$
> So the reason 4 to 5, is written thus, $4/5$.
> [...]
> $2/3 + 4/5 = 8/15$ This worke is done after the manner of multiplying fractions [Oxford, Bodleian Library, MS Aubrey 25, ff. 6v–7r].

This is baffling to the modern eye since the operations involved are virtually identical: to "add" (i.e., multiply) we multiply numerator by numerator and denominator by denominator; to subtract (i.e., divide) we cross-multiply numerators by denominators. But Mercator had a strong sense of the distinctiveness of ratio and of the important status of rationals. In *Rationes mathematicae* he said:

> For it is an old saying, that the minds of men are nothing other than number: whence thus in Music irrational and surd numbers, except insofar as they approach closely to rationals, are neither consonant with the progeny of unity nor are suitable to singing; but clatter in dissension and nothing but fights [Mercator, 1653, A2r–A2v].[10]

[9] Remember that this text seems to render the Latin *ratio* in English as *reason* rather than *ratio*. The antecedent of a ratio is its first term and the consequent its second.

[10] "Est enim vetus verbum, mentes humanas nihil aliud esse, quam numerum: unde quemadmodum in Musicis irrationales & surdi numeri, nisi quatenus ad rationales proxime accedunt, neq. consonant cum unitatis propagine, neq. ad cantum sunt idonei; sed dissensum & rixas meras crepant."

And in the rest of the pamphlet he explained some of the consequences of this distinction between rationals and irrationals.

Multiplication of a ratio was achieved initially by adding it to itself repeatedly, and Mercator then noted that, in effect, if we want to multiply a ratio by n we must raise both its antecedent and its consequent to the nth power [Oxford, Bodleian Library, MS Aubrey 25, f. 8v]. He was careful throughout to use the terminology specific to ratios: *duplicate* rather than *dupla* (the latter is an operation on numbers or magnitudes, the former on ratios), triplicate rather than tripla, etc. [see Sylla, 1984]. No equivalent in fractions was offered for this multiplication of a ratio by a number.

Next he observed that we can divide a ratio by a number by taking the appropriate root of both antecedent and consequent. Again he gave no equivalent operation for fractions, nor did he comment on the fact that this operation would in general leave us with a ratio of two irrationals. In fact Mercator only applied this method to ratios of two square or cube numbers, so that in his examples the answer was always rational [Oxford, Bodleian Library, MS Aubrey 25, f. 9v].

The explicit distinction between ratios and fractions occurs in no text other than "Musica"; and no other text leaves the introduction of logarithms so late. "Theoria musices" introduces logarithms immediately after the operations of addition and subtraction by ratios, and gives multiplication and division procedures by logarithms only; "Of music" introduces logarithms after the procedure of multiplication by repeated addition, and gives the division procedure by logarithms only. As with the ratio arithmetic, the exposition of logarithms is clearest in "Musica," and it begins much less abruptly than elsewhere, although "Notebook" does attempt to motivate it: "multiplication and division of ratios is effected very quickly by logarithms" [Oxford, Christ Church Library, MS. 1130, f. 5r].[11]

In "Musica" the material on logarithms is in four sections: how to find the measure of a ratio; how to multiply ratios; how to divide a ratio by a number; and how to divide one ratio or interval by another [Oxford, Bodleian Library, MS Aubrey 25, ff. 9v–12r]. This constitutes a complete solution to the problems of measuring and comparing musical ratios, and it is surprising that in this text Mercator did not go on to apply his mathematics to a specific system of tuning.

Mercator introduced logarithms in the same painstaking manner as his ratio calculations. First, "find the measure of the given reason in Logarithmes, which is donne by taking the Logarithmes of both the termes, and substracting the lesser Logarithm from the greater, for the remaynder is the measure of the given reason" [Oxford, Bodleian Library, MS Aubrey 25, ff. 9v–10r]. He gave an example. Next: to multiply a ratio by a number, multiply its measure by that number. He did not then take the antilogarithm in order to return from measures to ratios: in fact he could not, because the antilogarithm of a magnitude is a magnitude, not a ratio. What he did was to perform the same example calculation in ratios, and then take the logarithm of the answer: this gave the same result as the first, logarithmic calculation, which he took as proof that the first method worked [Oxford, Bodleian Library, MS Aubrey 25, f. 10r].

Division was explained, exampled, and "proved" (checked) similarly. Next he used multiplication and division to perform some musical comparisons: he showed that four musical fourths are less than three musical fifths "as 50 is lesse then 53." And "the tonus major exceedeth the sixth part of a Diapason, as 51 exceedeth 50." He also performed a subtraction, which operation he had not explained in the abstract, to show that the difference between a musical fourth and fifth is a whole tone: which, as he pointed out, is simple to compute by ratios in any case [Oxford, Bodleian Library, MS Aubrey 25, ff. 10v–11r]. Both 50 : 53 and 51 : 50 were arrived at by simply taking the first two significant figures of the calculated logarithms, so he had not yet quite computed a strict "ratio of ratios." This he did next.

> To divide one Musicall intervall by another. For instance, find how many times the tonus major (8/9) is conteyned in the diapason (1/2)
> Rule. Find the measure of both the reasons given, and divide the greater by the lesser... 0,051152 [/] 0,301030 [=] 5,88, so that in an Eighth there are conteyned 5 toni majores, and 88/100 of a tonus major [Oxford, Bodleian Library, MS Aubrey 25, f. 12r].

Here the material on logarithms in "Musica" ends. The idea was taken further in "Notebook." In fact the first example Mercator gave there of computing ratios using logarithms was a geometrical one: if two spheres have diameters in the ratio 2 : 3, what is the ratio of their volumes? For this problem he did need to take an antilogarithm after the relevant logarithm calculation, which gave him the answer 1 : 3.385. He made a slip: it should have been 1 : 3.375,

[11] "Multiplicatio et Division [*sic*] rationum magno compendio efficitur per Logarithmos."

but even so it is surprising he did not note that the equivalent and perhaps more informative answer 8 : 27 could have been found more easily by a straightforward ratio calculation [Oxford, Christ Church Library, MS. 1130, f. 5v].

He went straight on to compute the number of "commas" in an octave: 55.787. (The comma, here, was a musical interval with ratio 81 : 80, defined as the difference between a major third and two tones. Today it is known as the syntonic comma.) There follow the statements without solutions of a number of examples of adding or subtracting pairs of small ratios. Most of these can be related to divisions of the tetrachord given by Ptolemy, although it is not clear what the point would be of performing such simple computations by the method of logarithms. One possibility is that Mercator was trying out the use of various Greek small intervals as approximate common measures [Oxford, Christ Church Library, MS. 1130, f. 6r; Barker, 1989, 346–355].

Next he asked "whether six major tones may equal an octave," as though there were the possibility they could [Oxford, Christ Church Library, MS. 1130, f. 6r].[12] Again no working or solution is given. He suggested multiplying the comma by 55 and then by 56 to confirm that one result would be larger than an octave and the other smaller. Finally he asked how many commas were in a fourth and a fifth [Oxford, Christ Church Library, MS. 1130, ff. 6r–6v].

Next: "to find the ratio of ratios." "If the sizes of the octave and fifth are compared, that is 0.301030 and 0.176091, I say that the octave holds to the fifth (or the ratio 1/2 holds to 1/3) the ratio 53 to 31" [Oxford, Christ Church Library, MS. 1130, f. 6v].[13] Note that now any distinction between ratios and fractions has been abandoned. There the section on logarithms ends, except for some lengthy working relating to the calculation of this ratio 53 : 31. This ratio later motivated Mercator's division of the octave into 53 equal parts. He used the logarithms again, without further comment on the method itself, to list the measures ("mensurae") of the intervals generated by the division into 53 and also those of the meantone temperament [Oxford, Christ Church Library, MS. 1130, ff. 40r–47v].

Although details differed, Mercator's procedure in "Theoria musices" and "Of music" was broadly similar to that in "Notebook": he introduced the use of logarithms to measure intervals during the explanation of ratio arithmetic, gave examples, and showed how to divide an interval by a number or by another interval. He asked and answered questions about the number of tones in an octave, the number of commas in an octave and the ratio of a fifth to an octave. He used the latter to motivate a division of the octave into 53: the 53rds were thereafter called "artificial commas," because of their similarity to the "comma," mentioned above. He measured the intervals resulting from this division and compared them, using logarithms, with those of the just intonation and the meantone scale, two tunings often discussed in the 17th century. He advocated the meantone scale, for which the basic interval is a just fifth minus a quarter of a comma. It seems eminently reasonable to use logarithms to deal with this tuning, for which ratios are cumbersome. (This is the same scale that Mercator advocated in his text about Birchensha [London, British Library, Add MS. 4388, ff. 39–44].)

4.3. Implications

Mercator's insistence on the words "measure", "mensura", or "quantitas" for the logarithmic sizes of musical ratios is striking, as are his concern to calculate precise ratios of ratios and his evident interest in the logarithmic method for its own sake. For Mercator the logarithmic method was something more than a calculating device: it both solved a mathematical problem and provided a new way to conceive musical pitch.

In support of this we might expect Mercator to have made diagrams similar to Descartes', illustrating how logarithms transformed the musical string into the pitch line. In fact there are two such diagrams by Mercator in his

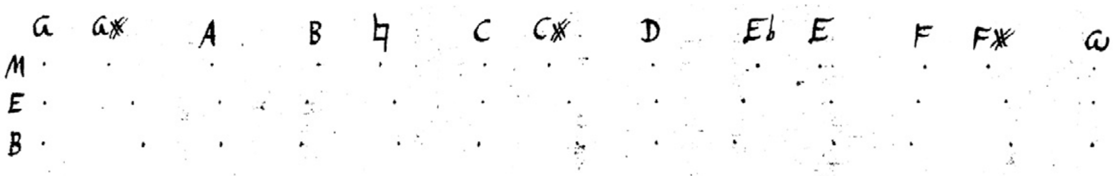

Fig. 3. Mercator's linear representation of pitch. ©The British Library. All Rights Reserved [MS Add. 4388, f. 44r].

[12] "Num sex toni majores aequipolleant octavae."

[13] "Invenire rationem rationum." "Si comparentur quantitates octavae et quintae, hoc est... 0,301030 et 0,176091; ajo octavam ad quintam (sive rationem 1/2 ad 2/3) obtinere rationem... 53 ad 31."

discussion of Birchensha's tuning in London, British Library, Add MS. 4388 (one is shown in Fig. 3). In the diagrams equal intervals are of equal sizes, as he was careful to explain: "I have drawn heare three lines, to represent the three severall Scales, [viz.] M mine, E the equall, and B Mr Berchinshaw's. Not to be understood, as three string[s] but by equall parts representing equall Musicall intervalls" [London, British Library, Add MS. 4388, ff. 43r, 44r].

This confirms explicitly what is only implied by Descartes' diagrams: that it was now possible to make diagrams in which pitch was quantified logarithmically, not by string length. Mercator's comparison of three tunings also gives a vivid sense that pitch is continuous, not discrete, in such a representation.

Although Kepler had already published explicit calculations of ratios of ratios using logarithms, Mercator was the first to apply them explicitly to music. His writings on the subject must have cost him considerable time and effort, and one wonders why he published nothing of them. The plague disrupted the *Rationes* project, which, had it been based on "Notebook" or material similar to it, would have presented the measurement and comparison of ratios together with extensive examples of their use in music, astronomy, geometry, and the making of calendars. I have no suggestion to offer about why this plan was never revived. Related material was passed to John Aubrey, Robert Hooke, and perhaps John Pell, but we have no evidence that it circulated any more widely. In *Logarithmo-technia*, Mercator's book on the construction of logarithms and infinite series, he stated that the use of logarithms was the measuring of ratio [Mercator, 1668, 1–2]. In *Hypothesis astronomica nova*, a study of the motions of the planets, he used logarithms in his example calculations. In *Logarithmo-technia* one passage does in fact hint at applying the logarithmic measure of ratios to musical ratios, but it does not spell out any of the consequences seen in the manuscripts [Mercator, 1668, 8–9].[14] When Holder mentioned Mercator's musical manuscripts in 1694, he focused on the division of the octave into 53 and did not describe the use of logarithms [Holder, 1694, 105–106].

Mercator's manuscripts also raise the question of his originality. *Rationes* was published in 1653, and I believe most of his musical manuscripts date from a similar period; and Descartes' *Compendium* was first published in 1650. This seems unlikely to be by chance: although Descartes did not describe a logarithmic method in his book there seems the strong possibility that Mercator arrived at the idea by studying Descartes' diagrams and noting their crucial property that equal intervals had equal sizes.

Mercator's division of the octave into 53, although linked in his manuscripts with the ratio he finds between the octave and the fifth, was surely a consequence of the appearance of such a division in Boethius. As Mercator conceived it, it was primarily an approximation scheme, a simpler equivalent of the logarithmic measure, based on the same mathematics: the arbitrarily precise scale from zero to log 2 was replaced by a 53-point scale on which all the points of interest could be approximated closely enough for (in some sense) practical purposes. The idea of using an equal division to measure intervals approximately was hardly a novelty by this period and would not distinguish Mercator from other theorists but for the logarithmic calculations that underlay it.

5. Newton

Isaac Newton's writings on music are not as extensive as Mercator's. The two most substantial pieces are an essay "Of Musick" in Cambridge, University Library, Add MS. 4000, ff. 137r–143v; and a set of musical calculations in Cambridge, University Library, Add MS. 4000, ff. 104r–113v and Add MS. 3958 (B), f. 31r (the latter appears to be a loose sheet from the former: it should possibly appear after f. 106). One sheet of the calculations is dated 20 November 1665 [Add MS. 4000, f. 105v].

"Of Musick" deals, broadly speaking, with the following topics: the keynote; octaves and their divisions into a fourth and a fifth, and into a third and a sixth; the grading of consonances and the use of dissonances; the progressive subdivision of intervals; the generating of modes from different orderings of the degrees of the scale; the modes' classification and grading in terms of "elegance"; the ordering of major and minor tones within these modes; solmization; and changes of key and mode, and which such changes are easiest. Newton noted that extra material needed to be inserted on the motion of strings, and on "logarithms of those strings, or distances of the notes" [Cambridge, University Library, Add MS. 4000, f. 137v].

The calculations in Cambridge, University Library, Add MS. 4000, ff.104–113 and Add MS. 3958 concern two subjects: the division of the octave using logarithms and the generation and classification of the modes. The latter is

[14] It is suggestive, though no more, that *Logarithmo-technia* and music were among the topics that Robert Hooke noted were discussed at the home of Sir Christopher Wren on 15 July 1676, by a group including Mercator, John Aubrey, and William Holder: see [Hooke, 1935/1968, 242].

a more detailed version of the similar material in "Of Musick" and it ends with a draft contents list that matches the contents of "Of Musick" quite closely. Here I am concerned only with the material on logarithms.

The table on f. 105v sheds the most light on what Newton was doing here; its column headings are as follows:

> [1] How the string 1 or 720 is to bee devided th[a]t it may sound all the musicall notes & halfe notes in an eight
> [2] The proportion w[hi]ch those musicall notes & 1/2 notes bear the one to the other (viz the logarithmes of the string sounding them[)]
> [3] Twelve exact or equidistant 1/2 notes (or the logarithmes of a cord divided into 12 geometricall partes) the distance of each 1/2 note being 0,025085833333 &c. A just note being 0,050171666666 &c
> [4] A string 720 divided into 12 (geometricall progressionall) parts, th[a]t it may sound the 12 exact 1/2 notes in an eight
> [5] The proportion of all the 12 Musicall 1/2 notes in an Eight; An exact halfe note being a Unite [Cambridge, University Library, Add MS. 4000, f. 105v]

So Newton computed the musical division of the octave in string lengths, for a string of length 1 or length 720, and then he took the logarithms of the string lengths. (His scale had the same set of intervals as the scale known as the just intonation, but rearranged to form a palindrome.) He also computed the string lengths for equal temperament and their logarithms. His terminology was different from Mercator's: he called the logarithmic measure the "distance" of an interval, and referred to the "proportions," not the ratios, that notes bore to one another.

In the final column he rescaled the list of logarithms so that they ran from 0 to 12, and therefore showed the size ("proportion") of each interval relative to the equal-tempered semitone. It has been noted that this anticipated the modern use of 1200 equally-tempered "cents" to measure the intervals in an octave [Gouk, 1999, 233–235].

Next in the manuscript (though not necessarily next in Newton's order of working) are calculations resulting in a table showing by what fraction of an equal-tempered semitone each note in his scale differed from the equivalent equal-tempered note. There are two apparently unrelated charts for finding the musical interval between any two notes of the scale (which Newton perhaps used later when listing the intervals in each mode) [Cambridge, University Library, Add MS. 4000, ff. 106r–108r].

Next there are several versions, two quite large, of a chart in which the notes of the just scale are correlated with their approximations in various different equal divisions of the octave. Altogether, the equal divisions that were tried out were those into 12, 20, 24, 25, 29, 36, 41, 51, 53, 59, 100, 120, and 612: the apparently most final version of the table shows those into 53, 612, 100, 36, 29, and 120 (in that order) [Cambridge, University Library, Add MS. 4000, ff. 108r–108v]. I do not know how these divisions were chosen: the manuscripts we have are clearly missing a good deal of working, which would possibly clarify this. One might expect of Newton an attempt to turn the logarithmic "distances" of the first table into approximate fractions, to use their denominators as possible equal divisions of the octave (in the same way that having found that the fifth is close to 31/53 of an octave, Mercator took up the division into 53), but this is not found here.

Finally there are two circular diagrams, drawn freehand but manifestly in imitation of Descartes' circular diagrams [Cambridge, University Library, Add MS. 4000, ff. 109r, 109v] (see Fig. 4). The first has six rings of solmization syllables, correlating scales on six different keynotes (Descartes' diagram, shown in Fig. 1, correlates scales on three different keynotes). Around the outside of the circle are two sets of numbers, giving the approximate positions of all the notes in terms of the divisions of the octave into 53 and 120. The second diagram has five sets of notes and shows the numbers for the division into 53 only. This completed Newton's consideration of musical logarithms as far as it survives.

The presence of diagrams clearly similar to those of Descartes leaves room for little doubt that Descartes' *Compendium* prompted Newton's work in these manuscripts: the remainder of Add MS. 4000 contains other Cartesian material and other work on logarithms. And I believe that the *Compendium* probably prompted the use of logarithms too: Newton noticed the equal-interval property of Descartes' diagrams and was able to reproduce it in his calculations by using logarithms. The idea of using logarithms to compare the sizes of the just intervals with their equal-tempered equivalents was an ingenious one, and it was logical to extend this to other equal divisions of the octave. This was a different development of the logarithmic method from that of Mercator, who seemed interested in the measurement and comparison of ratios for its own sake. But for Newton too logarithms were something more than a calculating tool, since he used them to create diagrams like Descartes' and to compute rather precisely the relative sizes of different intervals. By writing about the "proportion" that intervals bore to one another he too had made precise the concept of a ratio of (musical) ratios.

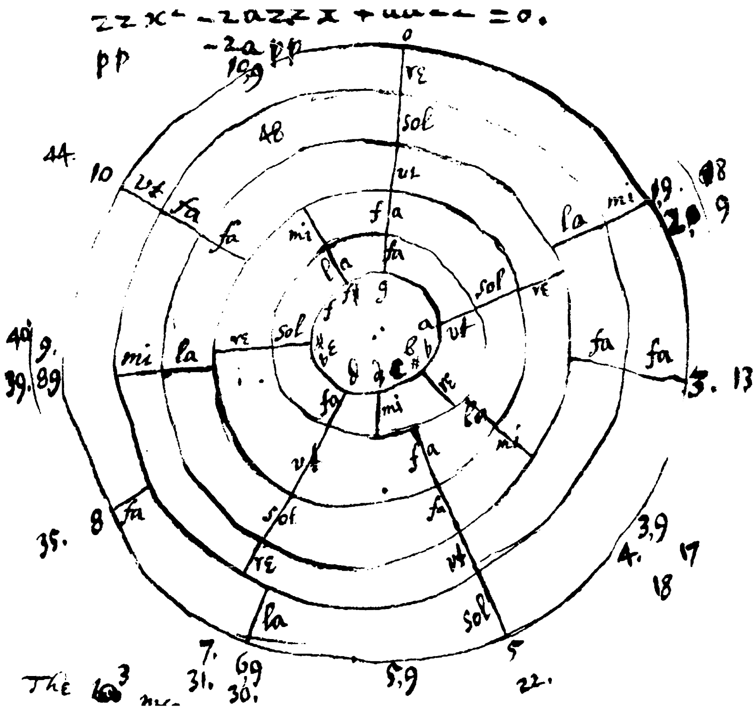

Fig. 4. Newton's circular representation of pitch [Cambridge, University Library, Add MS. 4000, f. 109r]. Reproduced by kind permission of the Syndics of Cambridge University Library.

Although he occasionally returned to musical topics later in his life Newton does not seem to have followed up the notion of ratios of ratios or of the measurement of ratios using logarithms. It does not seem useful to attempt to generalize about the complexities of his later uses of ratio, which bear no clear relation to musical ratios.

6. Conclusions

The concept of musical pitch and the measurement and comparison of musical intervals were problematic at least as early as Aristoxenus in the fourth century BC. Despite the domination of curricular music by the theory of ratios derived from Ptolemy via Boethius, there remained for at least some the intuition that musical intervals stood in well-defined ratios to one another and should be measurable in a way that reflected that.

With mathematical techniques available before the 17th century this intuition could not be made precise. Various responses to this problem existed, but an exact solution was impossible until the invention of logarithms, whose application to this particular musical problem originated with Descartes in about 1635.

Mercator and Newton may both, I suggest, have reasoned back from Descartes' diagrams to the unstated mathematical technique he used to produce them. Mercator made explicit the computation of ratios of ratios thus rendered possible and proposed to put the expanded theory of ratios to use in a variety of fields, while retaining a strong sense of the distinctiveness of ratio from magnitude. He could also retain the traditional basis of harmonic theory in ratios, computing where necessary the sizes and relative sizes of those ratios using logarithms. Newton was less explicit about ratios of ratios in the surviving manuscripts, but he too was aware that he had found a way to measure musical intervals and to make diagrams like Descartes'. So the invention of logarithms had facilitated a new quantification of pitch in which old questions about the relative sizes of intervals could be given exact answers. This had ramifications for the theory of music and of musical hearing that I have explored elsewhere [Wardhaugh, 2006, 177–210, 273–327; Wardhaugh, 2007]; it also illustrates something of the mathematical sophistication that studies of music could possess in the 17th century, a time when it was still possible to consider the study of music a part of mathematics.

References

Manuscripts

Cambridge, University Library, Add MS. 4000, ff. 137r–143v: Isaac Newton, essay “Of Musick”.

Cambridge, University Library, Add MS. 4000, ff. 104r–113v and Add MS. 3958 (B), f. 31r: Isaac Newton, musical calculations.

Groningen, University Library, MS. 108, ff. 60r–83v: copy of René Descartes, “Compendium musicae”, made for Frans van Schooten.

Leiden, University Library, MS. Hug. 29 a: copy of René Descartes, “Compendium musicae”, made for Constantijn Huygens.

London, British Library, Add MS. 4388, ff. 39–44: Nicolaus Mercator, commentary on the music theory of John Birchensha.

London, British Library, Add MS. 4388, ff. 70r–83v: copy of René Descartes, “Compendium musicae”, by John Pell.

London, Guildhall Library, MS. 51757 21, ff. 139–153: Nicolaus Mercator, “Mercator’s Musick”, copied for Robert Hooke.

Middelburg (Netherlands), Provincial Library of Zeeland, MS. “Journal of Beeckman”, ff. 163r–168v: copy of René Descartes, “Compendium musicae”, made by Isaac Beeckman.

Oxford, Bodleian Library, MS Aubrey 25, ff. 1–25: Nicolaus Mercator, “Musica Auctore N. Mercator 1673” (copied by Aubrey).

Oxford, Bodleian Library, MS. Aubrey 25, ff. 32–43: Nicolaus Mercator, “Musica 1672” (copied by Aubrey).

Oxford, Christ Church Library, MS. 1130: Nicolaus Mercator, mathematics and music notebook.

Oxford, Christ Church Library, MS. 1187 D 14: Nicolaus Mercator, “Theoria musices”.

Printed sources

Aristotle, 1984. Works: the Revised Oxford Translation (Jonathan Barnes, Ed.). Princeton University Press, Princeton.

Aristoxenus. Elementa harmonica. In: [Barker, 1989, 126–184].

Bailhache, P., 1992. Leibniz et la théorie de la musique. Klincksieck, Paris.

Barbour, J.M., 1933. The persistence of the Pythagorean tuning system. Scripta Mathematica 2, 286–304.

Barbour, J.M., 1940. Musical logarithms. Scripta Mathematica 7, 21–31.

Barbour, J.M., 2004. Tuning and temperament: A historical survey. Dover, New York. Reprint of the first edition of 1951.

Barker, A., 1978. Music and perception: A study in Aristoxenus. Journal of Hellenic Studies 98, 9–16.

Barker, A., 1989. Greek Musical Writings II: Harmonic and Acoustic Theory. Cambridge University Press, Cambridge, UK.

Barrow, I., 1734. The Usefulness of Mathematical Learning Explained and Demonstrated: Being Mathematical Lectures Read in the Publick Schools at the University of Cambridge (John Kirby, Trans.). Stephen Austern, London.

Benedetti, G.B., 1585. Diversarum speculationum mathematicarum et phusicarum liber. Nicolaus Bevilaqua, Turin.

Boethius, A.M.S., 1989. Fundamentals of Music (Calvin M. Bower, Trans., Claude V. Palisca, Ed.). Yale University Press, New Haven and London.

Burkert, W., 1972. Lore and Science in Ancient Pythagoreanism (E.L. Minar, Trans.). Harvard University Press, Cambridge, MA.

Burnett, C.F., Fend, M., Gouk, P.M., 1991. The Second Sense: Studies in Hearing and Musical Judgement from Antiquity to the Seventeenth Century. Warburg Institute, London.

Carpenter, N.C., 1958. Music in the Medieval and Renaissance Universities. Norman, University of Oklahoma Press, OK.

Christensen, T.S., 2002. The Cambridge History of Western Music Theory. Cambridge University Press, Cambridge, UK.

Cohen, H.F., 1984. Quantifying Music: The Science of Music at the First Stage of the Scientific Revolution, 1580–1650. Reidel, Dordrecht/Boston.

Descartes, R., 1650. Musicae compendium. van Zijll & Ackersdijck, Utrecht.

Descartes, R., 1653. Renatus Des-Cartes Excellent Compendium of Musick and Animadversions of the Author (Walter Charleton, Trans., William Brouncker, Ed.). Thomas Harper for Humphrey Moseley, London.
Descartes, R., 1656. Musicae compendium. Joannes Janssonius Junior, Amsterdam.
Descartes, R., 1668. Traité de la méchanique composé par Monsieur Descartes de plus l'abregé de musique du mesme autheur mis en François avec les éclaircissemens necessaires (N. Poisson, Trans.). Charles Angot, Paris.
Descartes, R., 1897–1909. Oeuvres de Descartes (C. Adam and P. Tannery, Eds.). Cerf, Paris.
Descartes, R., 1987. Abrégé de musique: compendium musicae (F. de Buzon, Ed.). Presses Universitaires de France, Paris.
Euclid. Sectio canonis. In: [Barker, 1989, 190–207].
Fend, M., 1991. The changing functions of senso and ragione in Italian music theory of the late sixteenth century. In: [Burnett et al., 1991, 199–221].
Field, C.D.F., Wardhaugh, B., 2008. John Birchensha: Writings on Music. Ashgate, Aldershot.
Gouk, P.M., 1999. Music, Science and Natural Magic in Seventeenth-Century England. Yale University Press, New Haven, CT/London.
Greengrass, M., Leslie, M., 1995/2002. Samuel Hartlib: The Complete Edition [of Sheffield University Library, Hartlib Papers; electronic edition on CD]. UMI, Ann Arbor; 2nd edition (revised and updated). Sheffield University, Sheffield, 2002.
Herlinger, J.W., 1981. Marchetto's division of the whole tone. Journal of the American Musicological Society 34, 193–216.
Holder, W., 1694. A Treatise on the Natural Grounds, and Principles of Harmony. J. Heptinstall for John Carr, London.
Hooke, R., 1935/1968. The Diary of Robert Hooke 1672–80 (Henry Robinson and Walter Adams, Eds.). Wykeham Publications, London.
Huygens, C., 1986. Le Cycle Harmonique (1691) and Novus cyclus harmonicus (1724) (Rudolph Rasch, Ed.). Diapason Press, Utrecht.
Kepler, J., 1619. Harmonices mundi libri quinque. Godofredus Tampachius, Linz. Also in: Kepler, J., The Harmony of the World (E.J. Aiton, A.M. Duncan, and J.V. Field, Eds.). American Philosophical Society, Philadelphia, 1997.
Kepler, J., 1624/1639. Chilias logarithmorum, praemissa demonstratione legitimaortus logarithmorum eorum[que] usus. Caspar Chemlinus, Marburg. Also in: Kepler, J., Gesammelte Werke, Vol. IX (W. von Dyck, M. Caspar, Eds.). C.H. Beck, Munich, 1938–, pp. 275–352. All citations are from the latter.
Kepler, J., 1625. Supplementum chiliadis logarithmorum. Caspar Chemlinus, Marburg. Also in: Kepler, J., Gesammelte Werke, Vol. IX (W. von Dyck, M. Caspar, Eds.). C.H. Beck, Munich, 1938–, pp. 353–426. All citations are from the latter.
Kircher, A., 1650. Musurgia universalis, sive ars magna consoni et dissone. Franciscus Corbellettus, Rome.
Malcolm, N., Stedall, J., 2005. John Pell (1611–1685) and His Correspondence with Sir Charles Cavendish: The Mental World of an Early Modern Mathematician. Oxford University Press, Oxford.
Marchetto de Padova, 1985. Lucidarium (J.W. Herlinger, Ed.). University of Chicago Press, Chicago.
Massa Esteve, M.R., 2003. La théorie euclidienne des proportions dans les *Geometriae speciosae elementa* (1659) de Pietro Mengoli. Revue d'Histoire des Sciences 56, 457–474.
Meibom, M., 1652. Antiquae musicae auctores septem. L. Elzevir, Amsterdam.
Mengoli, P., 1659. Geometriae speciosae elementa: ... Quartum de rationibus logarithmicis. Quintum de proprijs rationum logarithmis.... J.B. Ferronius, Bologna.
Mengoli, P., 1670. Speculationi di musica. Benacci, Bologna.
Mercator, N., 1653. Rationes mathematicae. P. Hakius, Copenhagen.
Mercator, N., 1668. Logarithmo-technia, sive, methodus construendi logarithmos nova.... W. Godbid, London.
Napier, J., 1614. Mirifici logarithmorum canonis descriptio. A. Hart, Edinburgh.
Neidhardt, J.G., 1724. Sectio canonis harmonici. C.G. Eckart, Königsberg.
Oresme, N., 1966. De proportionibus proportionum and Ad pauca respicientes (Edward Grant, Ed. and Trans.). University of Wisconsin Press, Madison, WI.
van Otegem, M., 1999. Towards a sound text of the Compendium musicae, 1618–1683, by Rene Descartes (1596–1650). Lias 26, 187–203.
Ptolemy. Harmonics. In: [Barker, 1989, 275–391].
Rossi, L., 1666. Sistema musico. La stampa episcopale, for Angelo Laurenzi, Perugia.

Saint-Vincent, Grégoire de, 1668. Opus geometricum posthumum ad mesolabium per rationum proportionalium novas proprietates. Balduinus Manilius, Ghent.

Sasaki, C., 1985. The acceptance of the theory of proportion in the sixteenth and seventeenth centuries: Barrow's reaction to the analytic mathematics. Historia Scientiarum 29, 83–116.

Sauveur, J., 1984. Collected Writings on Musical Acoustics (1700–1713) (Rudolf Rasch, Ed.). Diapason Press, Utrecht.

Scriba, C.J., 2004. Mercator, Nicolaus (1620?–1687). In: Oxford Dictionary of National Biography, vol. 37. Oxford University Press, Oxford, pp. 842–843.

Stevin, S., 1955–1966. On the Theory of the Art of Singing (A.D. Fokker, Trans.). In: Crone, E. (Ed.), Simon Stevin, The Principal Works of Simon Stevin, vol. V. C.V. Swets & Zeitlinger, Amsterdam, pp. 423–459.

Sylla, E., 1984. Compounding Ratios: Bradwardine, Oresme, and the first edition of Newton's Principia. In: Mendelsohn, Everett (Ed.), Transformation and Tradition in the Sciences, Essays in Honor of I. Bernard Cohen. Cambridge University Press, Cambridge, UK, pp. 11–43.

Wallis, J., 1682. Claudii Ptolemaei harmonicorum libri tres. At the Sheldonian Theatre, Oxford.

Wardhaugh, B. 2006. Mathematical and Mechanical Studies of Music in Late Seventeenth-Century England. Unpublished D.Phil. thesis, Oxford.

Wardhaugh, B., 2007. The Logarithmic Ear: Pietro Mengoli's mathematics of music. Annals of Science 64, 327–348.

Zarlino, G., 1558. Istitutioni harmoniche. Francesco Senese, Venice.